武汉大学 水资源与水电工程科学国家重点实验室
水科学讲坛第 8 讲, 2021 年 7 月 5 日

海洋超大型浮式结构物的水弹性响应
I: 移动载荷生成的水弹性波动

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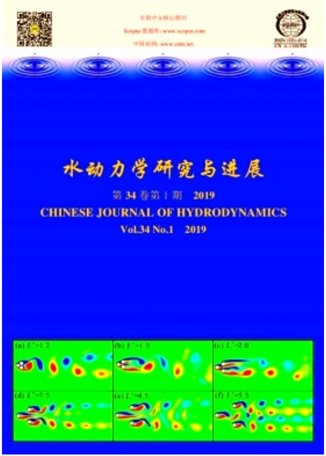
上海大学力学与工程科学学院
上海市应用数学和力学研究所

《水动力学研究与进展》(中文刊)[1]



水动力学研究与进展

第 34 卷第 1 期 2019

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Vol.34 No.1 2019

- 1986 年创刊, 双月刊;
- 中国科学引文数据库 (CSCD) 核心库
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- 编辑部真正的官网 http://www.jhydrodynamics.com

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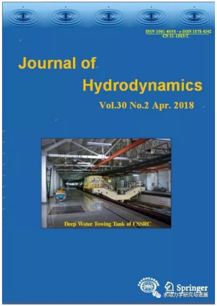
[1] "中国知网" 全文下载: https://navi.cnki.net/KNavi/JournalDetail?pcode=CJFD&pykm=SDLJ

Journal of Hydrodynamics, Springer[2][3][4]

Impact Factors in Journal Citation Report and **Rank** in Mechanics Category for JHD:

- 2018: 1.855 (Q3), 73/134
- 2019: 2.256 (Q2), 62/136
- 2020: **2.590** (Q2), 62/136



2012 年起, 连续 9 年评为"中国最具国际影响力学术期刊";
2019 年获"中国科技期刊卓越行动计划"支持.

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[2] Full articles in 2006–now: **https://link.springer.com/journal/42241**

[3] Journal information: **http://www.springer.com/journal/42241**

[4] Manuscript submission: **http://ijhd.editorialmanager.com**

学术交流平台

- 全国水动力学研讨会
  - 1986 至今，一年一届，已办 31 届 (2001 年，武汉)
  - 2021.10.29–11.1，无锡
- 海峡两岸水动力学研讨会
  - 2008 至今，两年一届，两岸轮值，已办 7 届
  - 2021，台湾
- 全球华人水动力学学术会议
  - 2006 与 2016 各办一届.
  - 第三届原定 2021 年无锡，因疫情延期
- International Conference on Hydrodynamics (ICHD)，国际水动力学学术会议
  - 1994 年首届中国发起，两年一届，至今已办 13 届.
  - 第 14 届原定 2020 在意大利罗马，因疫情延期

奖励

♠ 周培源水动力学奖

- “周培源基金会” 颁发证书与奖金
- “Journal of Hydrodynamics 编委会” 负责评选
- 周培源优秀水动力学论文奖, 1993–2010, 已颁 7 届
- 2011 年至今, 三年一届, 已颁 3 届; 2021 年第 4 届
- 一等奖、二等奖各一名, 青年奖 (40 岁以下) 两名

♠ Journal of Hydrodynamics 年度高被引论文奖

- “Journal of Hydrodynamics 编委会” 颁发证书与奖金
- 2015 至今, 每年一次, 研讨会开幕式颁发
- 前两年发表的, 在本年度被其他 SCI 期刊引用

♠ 全国水动力学研讨会学生优秀论文奖

- “Journal of Hydrodynamics 编委会” 和 “中国力学学会水动力学专业组” 联合评选并颁发证书与奖金
- 2015 起, 每年一届, 研讨会闭幕式颁发
- 学生第一作者, 且到会宣讲, 并全程参会

♡ **【教育经历】** ♡

- 复旦大学 学士 1995.7 应用力学系 力学专业
- 上海大学 硕士 1998.2 上海市应用数学和力学研究所
  流体力学 专业; 导师: 戴世强
- 香港大学 博士 2003.2 机械工程系 海洋与近海工程
  (流体力学) 专业; 导师: 章梓雄

♡ **【任职受聘】** ♡

- 上海大学 上海市应用数学和力学研究所
  研究实习员 (1998.3–1999.2)、讲师 (2003.3–2005.5)、
  副研究员 (2005.6–2009.1)、研究员 (2009.2–今)、
  博士生导师资格 (2009.5–今)
- 香港大学 机械工程系 Research Associate (2003–2007 期间累积 21 个月), Senior Research Associate (2012.8)

♡ **【国外访学】** ♡

- 加州大学伯克利分校 机械工程系 访问学者
  (2013.7–2014.6, 合作者: Ronald W. Yeung)

随“波”逐“流” 乘风破浪

♡ 重力波 ♡ 毛细重力波 ♡

- 非线性水波的 Hamilton 描述 (1995–1998)
- 内源兴波：潜航器遥感探测的水动力学基础
  - 粘性流体中的重力波 (1999–2007–今)
  - 粘性流体中的毛细重力波 (2006–2010–今)

♣ 水弹性波 / 挠曲重力波 ♣

- 背景一：极地区域的大冰层覆盖
- 背景二：海洋超大型浮式结构物
- 课题 1：运动载荷生成的水弹性波 (2006–今)
- 课题 2：波浪和结构物的相互作用 (2007–今)
- 课题 3：非线性水弹性波演化特性 (2011–今)
- 课题 4：水弹性振动的抑制 (2018–今)

♣ Methods:[5]

- PDE-based mathematical modeling / 数学模型
  - **Analytical, semi-analytical** solutions
  - Numerical solutions
    - ...
- Physical modeling / “物理模型”: Experimental approach
- Empirical formula for engineering problems / 工程中的经验公式

---

[5] -ical

工欲善其事，必先利其器

♡ 钱伟长的**三大法宝**:[6]

1. 张量分析
2. 渐近分析 / 摄动方法
3. 变分原理

♠ 研究方法:

- 继承以“(半) 解析”为主要研究手段
- 辅以数值计算、对照实验结果 (如有)

♣ 研究风格：林家翘的“应用数学过程” (3+1)

- 3: Formulation ⇒ Solution ⇒ Interpretation & Verification
- 1: Generation

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[6] 戴世强 (2003), “论钱伟长的治学理念和学术风格,” 力学进展, 33(1): 13

应用背景 / 前景

1. 人造的海洋超大型浮式结构物
2. 自然的极区大冰层、冰盖机场

Mega Float[7]

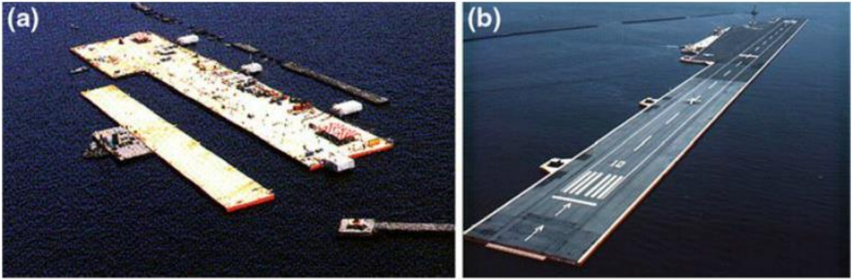

Fig. 2 Mega-float experimental models. **a** Phase I model. **b** Phase II model

The Technological Research Association of Megafloat (TRAM) was founded in 1995 with the support of the Ministry of Land, Infrasture, Transport and Tourism, Japan. TRAM conducted research and development on the Mega-Float until 2001.

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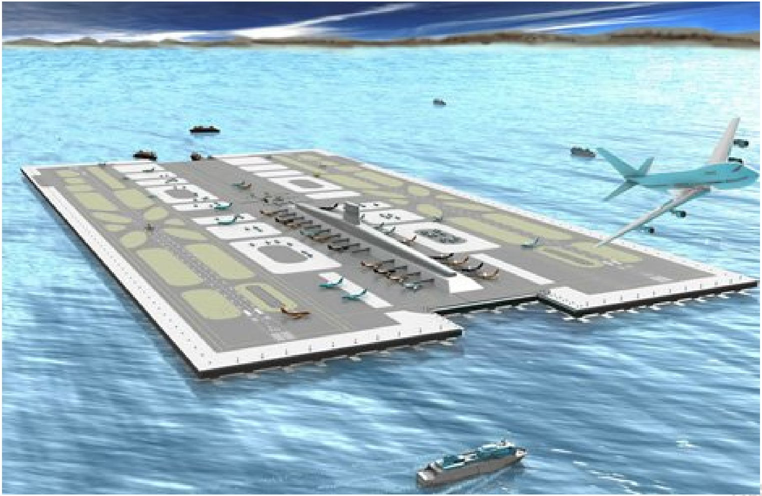
[7] M. Fujikubo & H. Suzuki (2015), “Mega-Float,” In: C. Wang & B. Wang (eds) “Large Floating Structures,” p. 198. Ocean Engineering & Oceanography, Vol. 3, Springer, Singapore.

海洋特殊军事基地



漂浮的圣地亚哥飞机场

2009 年 12 月 11 日 常州晚报

[http://epaper.cz001.com.cn/site1/czwb/html/2009-12/11/content_264903.htm]

超大型浮式结构物[8]

(Very Large Floating Structures)

定义

至少在一个方向上尺度达到公里级别的海洋工程结构物.

分类

- 厢式: 飘浮, 主要依靠浮力放置在海面上
- 半潜式: 支撑

统曰: “...... 若以大船小船各皆配搭, 或三十为一排, 或五十为一排, 首尾用铁环连锁, 上铺阔板, 休言人可渡, 马亦可走矣, 乘此而行, 任他风浪潮水上下, 复何惧哉? ” [《三国演义》第四十七回]

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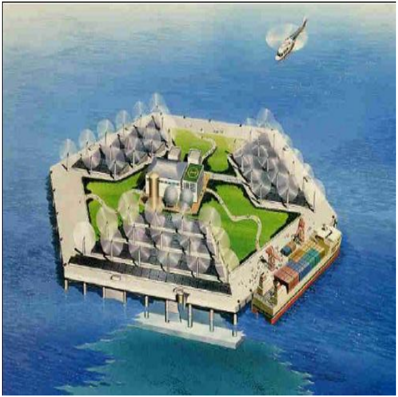
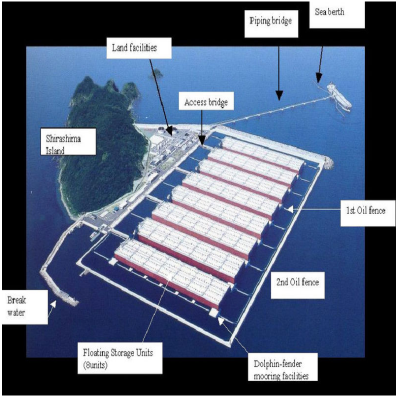
[8] 缪国平, 刘应中 (1996), “征服海洋之梦—超大型浮式海洋结构物,” 自然杂志, 18(1): 26–29.

VLFS 意义: 海洋资源开发



海上风能基地 (日本)

[E. Watanabe, C.M. Wang, T. Utsunomiya and T. Moan, 2004]



浮式储油基地 (Shirashima)

[www.wikiwaves.org/index.php/VLFS]

VLFS 意义：海洋空间利用

浮式应急基地

浮式机场 Haneda, Tokyo

[E. Watanabe, C.M. Wang, T. Utsunomiya and T. Moan, 2004]

Background (2):

Earth's Antarctic Ice Sheet[9]

Iceberg[10]

- 研究结果将为 VLFS 的设计、建造和维护提供科学的理论基础;
- 同时对于极地冰层动力学也有参考价值.

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[9] www.nasa.gov/multimedia/imagegallery/image_feature_576.html

[10] https://www.chaostrophic.com/antarctica-lose-1-biggest-icebergs-ever/

冰盖机场

中国首架极地固定翼飞机“雪鹰 601”在南极成功试飞
(2015 年 12 月 08 日 新华网)[11]
(2018 年 10 月 31 日 千龙网)[12]

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[11] http://gz.people.com.cn/n/2015/1208/c344113-27272081-4.html

[12] http://www.sohu.com/a/272422826_161623?_f=index_chan08news_7

冰盖机场[13]

- 2009 年，在中国第 25 次南极考察期间，我国曾在南极昆仑站以西约 3 公里处修建起长 4 公里、宽 50 米的“昆仑机场”跑道，用于固定翼飞机起降使用.
- 2010 年 1 月，我国第 26 次南极考察队又在南极内陆冰盖上再修建起一座简易机场“飞鹰机场”. 机场有长 600 米、宽 50 米的机场跑道，同时存放数百桶航空煤油，用于固定翼飞机紧急备降或加油补给.
- 2015 年，“雪鹰 601” 投入运行后，我国第 32 次南极考察队开始筹划一件大事 —在南极冰盖建永久机场. 机场位置位于距离中山站 28 公里的冰盖，跑道尺寸预计为 1500 米长 80 米宽，第 33 次南极考察队又在机场位置开展了测绘工作.
- 2018 年 10 月，第 35 次南极考察的一项重要任务，是要在南极冰盖开工建设我国第一个永久机场.

---

[13] 2018 年 10 月 28 日 科技日报 http://www.sohu.com/a/271808406_260616?_f=index_chan08news_3

Blue Ice Runways in Antarctica

NSF chartered Twin Otter on deck on wheels at Plunket Point, January 1989.

In the late 1980's NSF sponsored a project, with CRREL involvement, to locate "blue ice" runways–areas with no net annual snow accumulation, so that the resultant ice surface is capable of supporting aircraft landings using wheels instead of skis.[14]

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[14] http://www.southpolestation.com/trivia/history/blueice.html

Wilkins Ice Runway — Australian Antarctic Territory


Courtesy of Blackwood, Torsten, Fairfax

First passenger flight of the A319, with Minister for Environment and Heritage Peter Garrett onboard.

The 2.5-mile long by 330ft-wide runway was constructed on the inland plateau of the Upper Peterson Glacier, which is around 2,300ft thick and moves about 40ft each year. The service, which started in February 2008, uses an Airbus A319 jet.[15]

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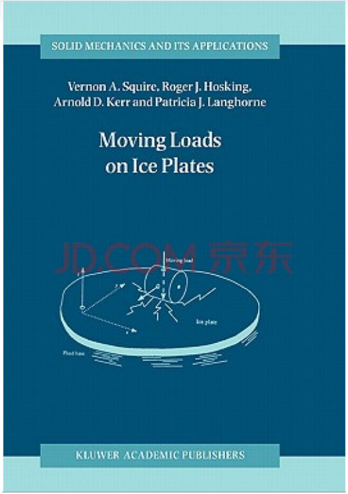
[15] http://www.airport-technology.com/projects/wilkins-ice-runway



SOLID MECHANICS AND ITS APPLICATIONS

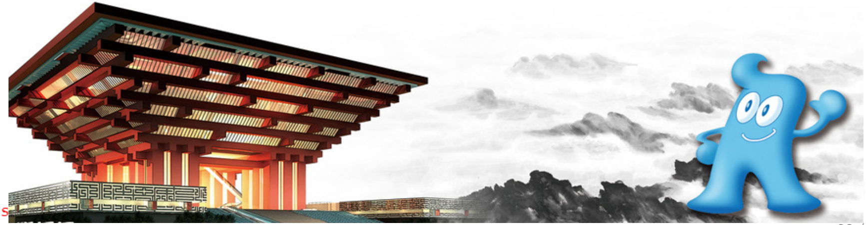
Vernon A. Squire, Roger J. Hosking,
Arnold D. Kerr and Patricia J. Langhorne

Moving Loads
on Ice Plates

KLUWER ACADEMIC PUBLISHERS

V. A. Squire, R. J. Hosking, A. D. Kerr, et al. (1996):
Moving Loads on Ice Plates.
The Netherlands: Kluwer Academic Publishers.

研究简况



-ics = ic + **science**:

- hydrostatics: 流体静力学
- hydrodynamics: 流体动力学，水动力学
- hydromechanics: 流体力学
- hydroelastic: 水弹性
- aeroelastics: 气动弹性学
- aerodynamics: 空气动力学
- aerostatics: 空气静力学[16]

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[16] http://fanyi.baidu.com

Hydroelasticity

Heller & Abramson (1959):[17] a concept

- “Hydroelasticity is defined as phenomena involving interactions among inertial, hydrodynamic, and elastic forces.”
- “The necessary and sufficient condition to classify a problem as one of hydroelasticity is that the elastic deformations of the structure induce additional hydrodynamic force.”

Bishop & Price (1979):[18] a formal discipline

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[17] S. R. Heller & H. N. Abramson (1959), “Hydroelasticity: A new naval science,” *Journal of the American Society for Naval Engineers*, 71(2): 205–209, DOI: 10.1111/j.1559-3584.1959.tb02326.x

[18] R. E. D. Bishop & W. G. Price (1979), “Hydroelasticity of Ships.” Cambridge University Press.

Hydroelasticity

自上世纪 80 年代以来，关于海洋超大型浮式结构物 (VLFS) 的研究蓬勃发展.

- 专业型会议：IWVLFS, ICHMT,[19] IWWWFB;[20]
- 综合型会议：OMAE, ISOPE, ICHD[21]......
- 国内：中国船舶科学研究中心、上海交通大学、陆军工程大学、大连理工大学、哈尔滨工程大学、石家庄铁道大学、中国科学院力学研究所、江苏科技大学、上海大学......

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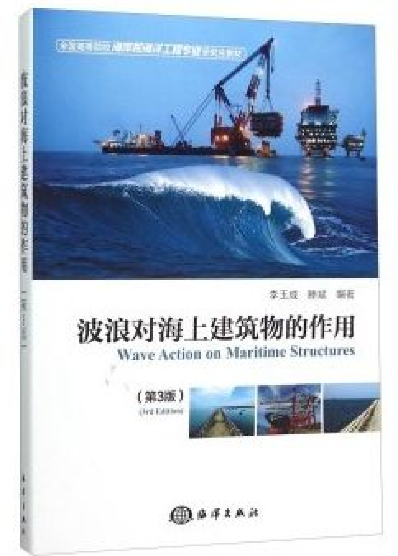
[19] The 8th International Conference on Hydroelasticity in Marine Technology, Seoul, Korea, Sep. 10–12, 2018; https://mhl.snu.ac.kr/hyel2018

[20] http://www.iwwwfb.org

[21] The 13th International Conference on Hydrodynamics, Songdo, Incheon Korea, Sep. 2–6, 2018; http://mhl.snu.ac.kr/ichd2018



李玉成, 滕斌 编著 第三版 (2015)[22]

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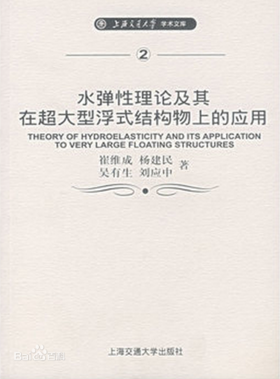
[22] § 2.9 波浪对水面半无限弹性板的作用

国内研究最新进展



上海交通大学 学术文库

②

水弹性理论及其
在超大型浮式结构物上的应用

THEORY OF HYDROELASTICITY AND ITS APPLICATION
TO VERY LARGE FLOATING STRUCTURES

崔维成 杨建民
吴有生 刘应中 著

上海交通大学出版社

崔维成, 杨建明,
吴有生, 刘应中:
“Theory of Hydroelasticity
and its Application to Very
Large Floating Structures,”
Shanghai Jiaotong University
Press (2007).

1. 吴有生, 田超, 宗智等, “近岛礁波浪环境下超大型浮式结构物的水弹性响应研究,” 第四届海峡两岸水动力学研讨会论文集, 1–17 页, 台北, 2013.10.28–11.3.[23]

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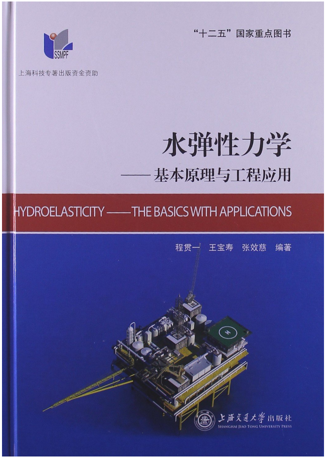
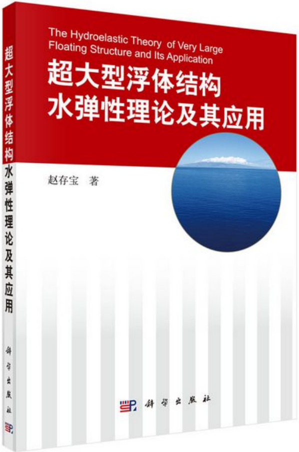
[23] 国家重点基础研究发展计划 (2013CB036100): “海洋超大型浮体复杂环境响应与结构安全性”.

国内研究最新进展



程贯一，王宝寿，张效慈 编著
(2013)



赵存宝 著
(2015)

研究热点：载荷作用下浮式结构物的水弹性响应

表面的冲击和滑行

[Z. Gao, 2008]

- 我们的工作：集中载荷诱发的远场挠曲重力波；

周边波浪和流动

- 我们的工作：单 / 双 / 多层流体中线性波浪与半无限 / 单或多个有限漂浮薄板相互作用

Our previous **analytical** work on hydroelastic waves [24]

♠ Linear problems:–

1. Generation of hydroelastic (flexural–gravity) waves
    1. Green's Function Method
    2. Integral Transform
    3. Local Asymptotic Analysis — Methods of Stationary Phase and of Steepest Descent
2. Interaction of incident gravity waves with VLFS
    - Method of Matched Eigenfunction Expansion

---

[24] English version for the International Summer School on Naval Architecture, Ocean Engineering and Mechanics, Shanghai Jiao Tong University, 2016–2019.

Our previous **analytical** work on hydroelastic waves

♠ Nonlinear problems:–

1. Propagation of nonlinear progressive waves
   - Homotopy Analysis Method (HAM)[25]

2. Head-on collision of two solitary waves
   - Global Asymptotic Analysis
     — Singular Perturbation Methods
     — Poincaré–Lighthill–Kuo (PLK)[26] method of strained co-ordiniates

---

[25] Proposed in 1992 by Shi-Jun Liao

[26] Jules Henri **P**oincaré (1854.4.29–1912.7.17); Michael James **L**ighthill (1924.1.23–1998.7.17); Yung-Huai **K**uo (1909.4.4–1968.12.5)

Our previous work on linear hydroelastic waves: I-(1)

Generation of hydroelastic waves by moving loads:

- D. Q. Lu* & S. Q. Dai (2006), Archive of Applied Mechanics, 76: 49–63.
- D. Q. Lu* & S. Q. Dai (2008), International Journal of Engineering Science, 46: 1183–1193.
- D. Q. Lu* & S. Q. Dai (2008), Acta Mechanica Sinica, 24(3): 267–275.
- D. Q. Lu* & H. Zhang[27] (2013), Theoretical Applied Mechanics Letters, 3(2): 022002.
- D. Q. Lu* & C. Z. Sun[28] (2013), Journal of Hydrodynamics, 25(3): 339–347.
- H. Zhang & D. Q. Lu* (2013), Chinese Journal of Hydrodynamics, 28(5): 615–625 (in Chinese).

---

[27] Master student (2009) in Applied Mathematics

[28] Master student (2010) in Applied Mathematics

Our previous work on linear hydroelastic waves: I-(2)

Generation of hydroelastic waves by moving loads:

- D. Q. Lu (2014), Journal of Hydrodynamics, 26(2): 339–341.
- D. Q. Lu* & R. W. Yeung (2015), International Journal of Offshore and Polar Engineering, 25(1): 8–12.
- J. S. Li[29] & D. Q. Lu* (2017), Journal of Hydrodynamics, 29(6): 1000–1009.
- D. Q. Lu (2020), Journal of Hydrodynamics, 31(5): 1024–1027.

---

[29] Master student (2014) in Fluid Mechanics

Our previous work on linear hydroelastic waves: II-(1)

Interaction of incident gravity waves with VLFS:

- F. Xu[30] & D. Q. Lu* (2009), Journal of Hydrodynamics, 21(4): 526–530.
- F. Xu & D. Q. Lu* (2010), International Journal of Engineering Science, 48(9): 408–419.
- F. Xu & D. Q. Lu* (2011), SCIENCE CHINA Physics, Mechanics & Astronomy, 54(1): 59–66.
- Q. Lin[31] & D. Q. Lu* (2013), Applied Ocean Research, 43: 71–79
- Q. Lin & D. Q. Lu* (2014), European Journal of Mechanics B/Fluids, 44: 10–21.
- Q. Lin, D. Q. Lu* & R. W. Yeung (2014), China Ocean Engineering, 28(5): 671–687.

---

[30] Master student (2007) in Fluid Mechanics

[31] Master student (2011) in Fluid Mechanics

Our previous work on linear hydroelastic waves: II-(2)

Interaction of incident gravity waves with VLFS:

- Q. R. Meng[32] & D. Q. Lu* (2015), Procedia Engineering, 126: 270–274.
- Q. R. Meng & D. Q. Lu* (2017), Journal of Fluids and Structures, 68: 295–309.
- Q. R. Meng & D. Q. Lu* (2017), Applied Mathematics and Mechanics — English Edition, 38(4): 567–584.
- Q. R. Meng & D. Q. Lu* (2018), European Journal of Mechanics B/Fluids, 67: 329–340.
- J. Pu[33] & D. Q. Lu* (2019), Chinese Journal of Theoretical and Applied Mechanics, 51(6): 1614–1629 (in Chinese).
- Y. G. Gong[34], D. Q. Lu* & J. Pu (2021), Journal of Harbin Engineering University, 42(7): 967–974 (in Chinese).

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[32] Master student (2013) in Fluid Mechanics

[33] Master student (2017) in Fluid Mechanics

[34] Undergraduate student (2016–2020) in Mechanics

Our previous work on nonlinear hydroelastic waves

♡ Propagation of progressive waves:

- P. Wang[35] & D. Q. Lu* (2013), SCIENCE CHINA Physics, Mechanics & Astronomy, 56(11): 2170–2177.
- P. Wang & D. Q. Lu* (2016), Applied Mathematics and Computation, 274: 700–710.
- P. Wang & D. Q. Lu* (2021), Advances in Applied Mathematics and Mechanics, 13(3): 724–734.

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[35] PhD student (2011) in Fluid Mechanics

Our previous work on nonlinear hydroelastic waves

♡ Head-on collision of two solitary waves:

- M. M. Bhatti[36] & D. Q. Lu* (2018), Qualitative Theory of Dynamical Systems, 17(1): 103–122.
- M. M. Bhatti & D. Q. Lu* (2018), Chinese Journal of Theoretical and Applied Mechanics, 50(6): 1406–1417 (in Chinese) with the English version in Theoretical and Applied Mechanics Letters, 8(6): 384–392.
- M. M. Bhatti & D. Q. Lu* (2019), Symmetry, 11(3): 333.
- M. M. Bhatti & D. Q. Lu* (2019), Open Physics, 17(1): 177–191.
- M. M. Bhatti & D. Q. Lu* (2019), Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 233(17): 6135–6148.

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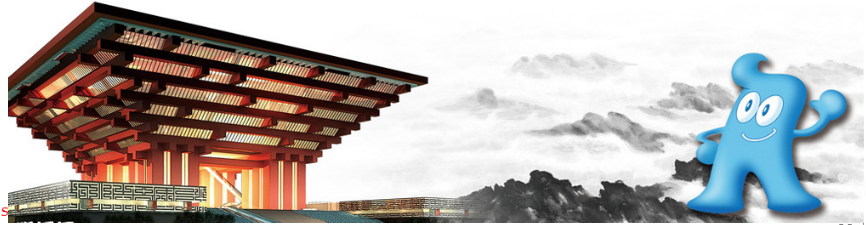
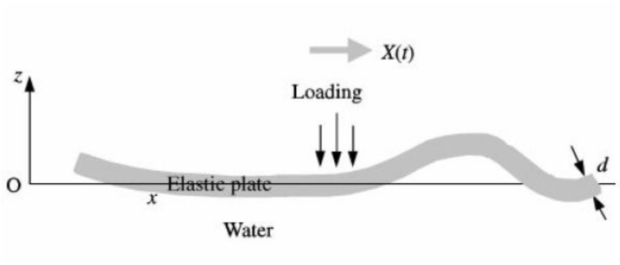
[36] PhD student (2014) in Applied Mathematics

Review



Moving load: theoretical modeling

For example,



- R. W. Yeung & J. W. Kim (2000), "Effects of a translating load on a floating plate — Structural drag and plate deformation," *Journal of Fluids and Structures*, 14: 993–1011.

Model for the structure: A thin elastic plate

An inviscid fluid covered by a thin elastic plate is a simple mathematical model for

- the very large floating structures (VLFS) in the offshore region,
- the ice cover in the polar region.

Thus there are synergies between

- VLFS hydroelasticity and
- sea ice research,

as demonstrated by Squire (2008).[37]

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[37] V. A. Squire (2008), International Journal of Offshore and Polar Engineering, 18: 241–253.

Model for the ocean: One- and two-layer fluid

♣ A simplest model is traditionally employed as a starting point:

- a single layer fluid

♣ As is well known, the ocean is stratified in density due to the vertical variation of water temperature and salinity. A simple but useful model for the ocean is

- a two-layer fluid model,[38][39]

in which the thin pycnocline is represented by a sharp interface.

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[38] R. W. Yeung & T. C. Nguyen (1999), Journal of Engineering Mathematics, 35: 85–107.

[39] M. R. Alam, Y. M. Liu & D. K. P. Yue (2009), Journal of Engineering Mathematics, 65: 179–200.

Multiple-layer model for the ocean

- a three-layer fluid model,[40][41][42]
- M-layer model.[43]

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[40] R. Mondal & T. Sahoo (2014), Zeitschrift für angewandte Mathematik und Physik, 65: 349–375.

[41] J. Grue (2015), Proceedings of the 30th International Workshop on Water Waves and Floating Bodies, Bristol, UK, 12–15 Apr. 2015.

[42] Q. R. Meng & D. Q. Lu* (2017), Applied Mathematics and Mechanics - English Edition, 38(4): 567–584, DOI: 10.1007/s10483-017-2185-6.

[43] Q. R. Meng & D. Q. Lu* (2018), European Journal of Mechanics B/Fluids, 67: 329–340, DOI: 10.1016/j.euromechflu.2017.09.010.

The present work: Objective

♠ Structure:

To consider the effects of the compressive (lateral) stress ($Q \neq 0$) of a beam/plate on hydroelastic responses:-

- wave profiles,
- wave resistance.

♠ Fluid:

- a single-layer fluid;
- a two-layer model.

♠ Concentrated load:

- at a fixed point;
- with a moving speed.

The present work: Objective

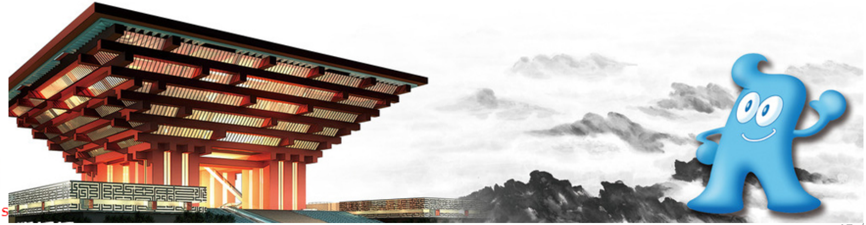
♠ Fixed loads:

1. an instantaneous load
2. an oscillating load
3. with a uniform current

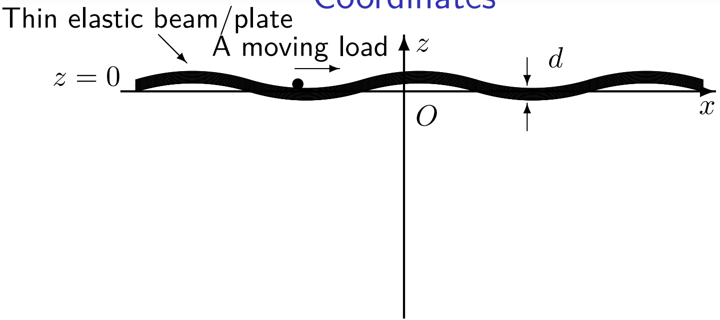
♠ Moving loads:

1. a steadily translating load
2. an impulsively starting load
3. a suddenly stopping load

Mathematical Formulation

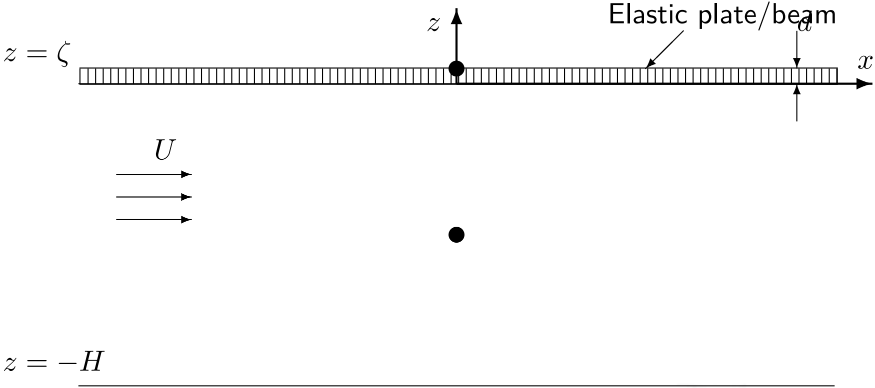


Coordinates



- A concentrated load moving on the beam/plate
- A thin elastic beam/plate of infinite extent floating on the fluid
- An inviscid, incompressible and homogeneous fluid of finite/infinite depth

Mathematical formulation



Governing equation

The governing equation is

$$\nabla^2\Phi = S_0\delta(\boldsymbol{x}-\boldsymbol{x}_0)f_0(t), \qquad (1)$$

where

- $\Phi(\boldsymbol{x}, t; \boldsymbol{x}_0)$ is the velocity potential for the perturbed flow,
- S_0 the constant strength of the simple mass source located at $\boldsymbol{x}_0$;
- $\boldsymbol{x}$ an observation point, t the time.
- $\delta(\cdot)$ the Dirac delta function.
- $f_0(t)$ a function of t.

The linearized kinematical and dynamical conditions at $z = 0$ are given by

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} - \frac{\partial \Phi}{\partial z} = 0, \tag{2}$$

$$\rho\left(\frac{\partial \Phi}{\partial t} + U\frac{\partial \Phi}{\partial x} + g\zeta\right) + D\nabla^4\zeta + Q\nabla^2\zeta + M\frac{\partial^2 \zeta}{\partial t^2}$$
$$= -P_0\delta(\boldsymbol{z} - \boldsymbol{z}_0)f(t) \equiv P_{\mathrm{ext}}, \tag{3}$$

where

- $D = Ed^3/[12(1-\nu^2)]$ be the flexural rigidity,
- Q is related to the lateral stress of the plate

  $Q > 0$: compression;

  $Q < 0$: stretch
- $M = \rho_{\mathrm{e}} d$ is the mass of the plate.

Capillary–gravity waves

In particular, as $D = 0$ and $Q = -T$, Eq. (3) is for the capillary–gravity waves on

- an inertial surface $(M \neq 0)$
- a free $(M = 0)$ surface

where T with $T > 0$ is the coefficient of the surface tension.

The dynamic boundary condition on $z = 0$ reads

$$\rho\left(\frac{\partial\Phi}{\partial t} + U\frac{\partial\Phi}{\partial x} + g\zeta\right) - T\nabla^2\zeta + M\frac{\partial^2\zeta}{\partial t^2} = -P_0\delta(\boldsymbol{z} - \boldsymbol{z}_0)f(t). \tag{4}$$

The bottom condition at $z = -H$ is given by

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z=-H} = 0, \tag{5}$$

where H is a positive constant.

It is assumed that the entire fluid is at rest for $t < 0$. Therefore, the initial conditions at $z = 0$ are

$$\Phi|_{t=0} = -\frac{I_0}{\rho}\delta(\boldsymbol{z} - \boldsymbol{z}_0), \tag{6}$$

$$\zeta|_{t=0} = E_0\delta(\boldsymbol{z} - \boldsymbol{z}_0), \tag{7}$$

$$\left.\frac{\partial \zeta}{\partial t}\right|_{t=0} = 0. \tag{8}$$

Some Dispersion Relations; Method of Solution

Dispersion relation for a single fluid with $\gamma = 0$

For the flexural waves on the elastic beam/plate floating on the invicid fluid of finite depth,

$$\omega^2 = \frac{gk(\Gamma k^4 - \Lambda k^2 + 1)}{\coth(kH) + \sigma k}. \tag{9}$$

$$\Gamma = D/\rho g, \qquad \Lambda = Q/\rho g, \qquad \sigma = M/\rho, \tag{10}$$

Special cases:–

1. $\Lambda = 0$ [44][45]
2. $H \to \infty$ [46]

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[44] D. Q. Lu* & S. Q. Dai (2006), Archive of Applied Mechanics, 76: 49–63, DOI: 10.1007/s00419-006-0004-1.

[45] D. Q. Lu* & S. Q. Dai (2008), International Journal of Engineering Science, 46: 1183–1193, DOI: 10.1016/j.ijengsci.2008.06.004.

[46] D. Q. Lu (2015), Proceedings of the 30th International Workshop on Water Waves and Floating Bodies, pp. 129–132, Bristol, UK.

Hydroelastic waves on the surface of a two-layer fluid

The dispersion relation for the flexural–gravity waves in the plate-covered region in a two-layer fluid ($\gamma \neq 0$) of finite depth[47], for $m = 1, 2$,

$$\omega^2 = \frac{gk(G_1 t_1 + G_2 t_2 + \varepsilon\sigma t_1 t_2)}{2[1 + \gamma t_1 t_2 + \sigma(t_1 + \gamma t_2)]} \times \left\{ 1 + (-1)^{m+1} \sqrt{1 - 4\varepsilon \frac{G_1 t_1 t_2 [1 + \gamma t_1 t_2 + \sigma(t_1 + \gamma t_2)]}{(G_1 t_1 + G_2 t_2 + \varepsilon\sigma t_1 t_2)^2}} \right\} \tag{11}$$

where $t_1 = \tanh kh_1$, $t_2 = \tanh kh_2$, $h_2 = H - h_1$.

$\gamma = \rho_1/\rho_2$, $\varepsilon = 1 - \gamma$,

$\Gamma = k^4 D/(\rho_1 g)$, $\sigma = kd\rho_e/\rho_1 = kM/\rho_1$,

$G_1 = 1 + \Gamma$, $G_2 = 1 + \gamma\Gamma$.

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[47] D. Q. Lu* & C. Z. Sun (2013), Journal of Hydrodynamics, 25(3): 339–347, DOI: 10.1016/S1001-6058(11)60372-8.

Gravity wave on the free surface of a two-layer fluid

As the thickness of the plate d tends to zero, Eq. (11) reduces to Eq. (12) obtained by Yeung & Nguyen (1999),[48]

$$\omega^2 = \frac{gk(t_1 + t_2)}{2(1 + \gamma t_1 t_2)} \left[1 + (-1)^{m+1} \sqrt{1 - 4\varepsilon \frac{t_1 t_2 (1 + \gamma t_1 t_2)}{(t_1 + t_2)^2}} \right]. \tag{12}$$

As the density ratio γ tends to zero, Eq. (12) reduces to the well-known dispersion relation for a single fluid:

$$\omega^2 = gk \tanh(kH). \tag{13}$$

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[48] R. W. Yeung & T. C. Nguyen (1999), Journal of Engineering Mathematics, 35: 85–107.

The dispersion relation for the flexural–gravity waves in a two-layer fluid with a uniform current:[49]

$$\frac{(\omega_m - \omega_c)^2}{\omega_0^2} = \frac{B}{2A}\left[1 + (-1)^{m+1}\sqrt{1 - \frac{4AC}{B^2}}\right], \tag{14}$$

where

$$\begin{aligned}
A &= 1 + \gamma t_1 t_2 + G_3(t_1 + \gamma t_2),\\
B &= G_1 t_1 + G_2 t_2 + \varepsilon G_3 t_1 t_2,\\
C &= \varepsilon G_1 t_1 t_2,\\
G_1 &= \Gamma k^4 - \Lambda k^2 - \chi \alpha^2 + 1,\\
G_2 &= \gamma G_1 + \varepsilon,\\
G_3 &= \sigma k,
\end{aligned}$$

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[49] D. Q. Lu (2014), Journal of Hydrodynamics, 26(2): 339–341, DOI: 10.1016/S1001-6058(14)60037-8.

Dispersion relation with $U \neq 0$

$$\Gamma = D/(\rho_1 g),$$
$$\Lambda = Q/(\rho_1 g),$$
$$\sigma = M/\rho_1,$$
$$\chi = U^2\sigma/g,$$
$$\omega_c = U\alpha,$$
$$\gamma = \rho_1/\rho_2,$$
$$\varepsilon = 1 - \gamma,$$
$$t_1 = \tanh kh_1, \qquad t_2 = \tanh kh_2,$$

and

$$\omega_0 = \sqrt{gk}.$$

As $U = 0$, we have a simplified form.[50]

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[50] D. Q. Lu* & C. Z. Sun (2013), Journal of Hydrodynamics, 25(3): 339–347, DOI: 10.1016/S1001-6058(11)60372-8.

- This is the end of the mathematical formulation & method of solution for Part 1: Hydroelatic wave generation.

- Let us move to Part 2: Hydroelatic interaction of waves with a floating plate.

武汉大学 水资源与水电工程科学国家重点实验室
水科学讲坛第 8 讲，2021 年 7 月 5 日

海洋超大型浮式结构物的水弹性响应
II: 波浪与结构物相互作用

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上海市应用数学和力学研究所

1. 数学描述：线性模型
2. 求解方法：色散关系，本征展开
3. 研究内容
4. 主要亮点

数学描述：线性模型

♠ 线性模型

- 强分层流体：两层，三层，N 层
- 结构小挠度：Euler–Bernoulli 梁，Kirchhoff–Love 板

♠ 非线性模型

- 流体运动非线性：Euler 方程 $\Longrightarrow$
    - 浅水长波：Boussinesq 模型
    - 中等深度：Nwogu's Boussinesq 模型 (1993)
- 结构振动非线性
    - 几何非线性：Plotnikov–Toland 板 (2008, 2011)

海洋垂向结构: 密度层结[1]

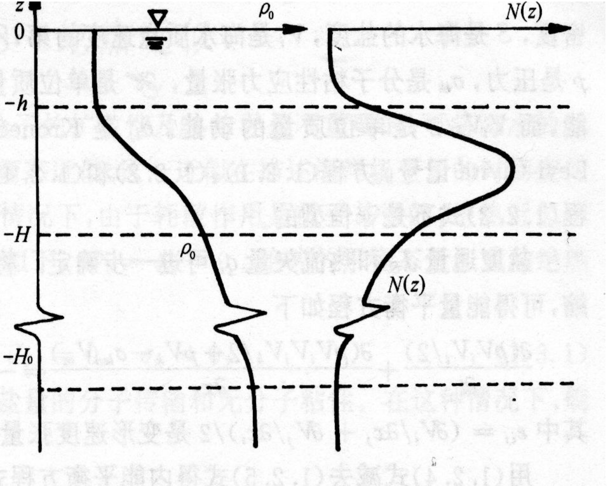

图 1-1 海洋中的典型 $\rho_0(z)$ 分布及 $N(z)$ 分布示意图

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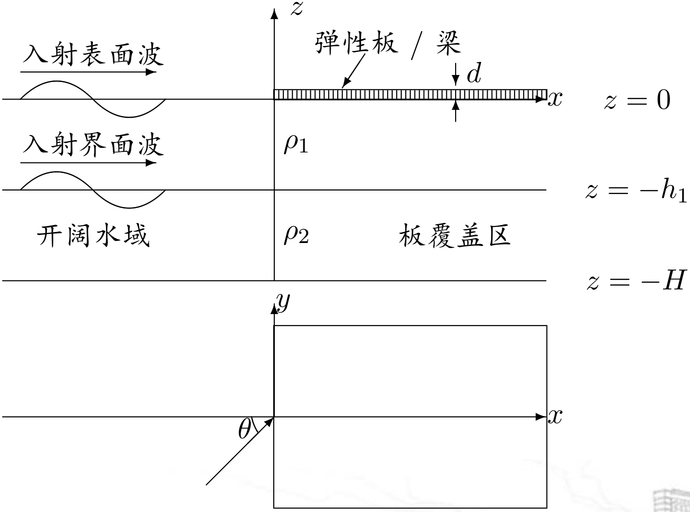
[1] 徐肇廷《海洋内波动力学》第 3 页, 科学出版社, 1999.

我们的工作: 水波与水平结构物的相互作用

- **流体**: 不可压, 无粘; 无旋; 有限深
  - 单层
  - 两层、三层
  - M 层均质
- **水波**: 单频谐波, 小振幅
  - 表面波
  - 界面波
- **结构**: 弹性薄梁 / 板, 小挠度
  - 单模块 (有限长、半无限长、圆盘)、
  - N 模块
- **作用**:
  - 正入射: 二维
  - 斜入射: 准二维

举例：模型示意图[2]



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[2] Q. Lin & D. Q. Lu* (2013), Applied Ocean Research, 43: 71–79, DOI: 10.1016/j.apor.2013.07.009.

数学模型：控制方程 (共性)+ 边界条件 (个性)

1. 流体区域：控制方程
2. 上表面边界条件 时域问题中运动学和动力学的组合
   - 开阔区域的自由表面
   - 板盖区域的弹性表面
3. 上下层流体的内界面
   - 法向速度的连续性
   - 法向压力的连续性
   - 运动学和动力学的组合边界条件
4. 平坦底部边界条件：无渗透
5. 板边缘条件：自由端，简支端，固支端
6. 开阔区域和板盖区域的速度和压力的连续性衔接条件(垂向)

数学模型：流体部分

假设:

- 流体：有限深，两层均质，不可压，无粘
- 运动：无旋，时谐.

流体速度势写成

$\phi_m(x,y,z,t) = \mathrm{Re}[\Phi_m(x,y,z)\mathrm{e}^{-\mathrm{i}\omega t}]$,

其中 $m=1,2$ 分别对于上下层流体. 则控制方程

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right)\Phi_m = 0,$$

$$(-\infty < x < \infty, \quad -H < z < 0), \tag{1}$$

式中 $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

自由表面

假设:

- 单频谐波，小振幅 $\Rightarrow$ 线性的表面波和界面波.

自由面上的边界条件 $(z=0)$ 的时域方程:

$$\frac{\partial \zeta}{\partial t}=\frac{\partial \Phi_1}{\partial z}, \qquad (\text{运动学}), \qquad (2)$$

$$\frac{\partial \Phi_1}{\partial t}+\frac{P_{\mathrm{e}}}{\rho_1}+g\zeta=0, \qquad (\text{动力学}). \qquad (3)$$

对于频域问题，可得组合的边界条件 $(P_{\mathrm{e}}=0)$:

$$-\omega^2\Phi_1+g\frac{\partial \Phi_1}{\partial z}=0, \quad (-\infty<x<0,\ z=0). \qquad (4)$$

数学模型：结构物（板／梁）

假设

1. 流体与板的下表面不分离
2. 弹性板的吃水为零
3. 板是纯弯曲，不存在刚体运动

弹性小挠度薄板的垂向运动 ζ 满足薄板微分方程 $(z=0)$

$$P_\mathrm{e} = D\nabla^4\zeta + Q\nabla^2\zeta + M\left(\frac{\partial^2\zeta}{\partial t^2} + g\right), \tag{5}$$

式中 $D = Ed^3/12(1-\nu^2)$，$M = \rho_\mathrm{e} d$.

数学模型：流固耦合

流固耦合的线性动力学边界条件：

$$\frac{\partial \Phi_1}{\partial t} + g\zeta + \frac{1}{\rho_1}\left[D\nabla^4\zeta + Q\nabla^2\zeta + M\left(\frac{\partial^2\zeta}{\partial t^2} + g\right)\right] =$$

将板的挠曲方程和流体的运动学和动力学边界条件结合起来，得

时域上：

$$\rho_1\frac{\partial^2\Phi_1}{\partial t^2} + \left(D\nabla^4 + Q\nabla^2 + M\frac{\partial^2}{\partial t^2} + \rho_1 g\right)\frac{\partial \Phi_1}{\partial z} = 0.$$

频域上：

$$-\rho_1\omega^2\Phi_1 + \left(D\nabla^4 + Q\nabla^2 - M\omega^2 + \rho_1 g\right)\frac{\partial \Phi_1}{\partial z} = 0.$$

界面边界条件

连续性：上下两层流体垂向速度、压力

♡ 界面上 $(-\infty < x < +\infty,\ z = -h_1)$ 组合的运动学和动力学条件及连续性条件，可得

$$\frac{\partial \Phi_1}{\partial z} = \frac{\partial \Phi_2}{\partial z} = -\mathrm{i}\omega\eta, \tag{6}$$

$$\gamma\left(\frac{\omega^2}{g}\Phi_1 - \frac{\partial \Phi_1}{\partial z}\right) = \frac{\omega^2}{g}\Phi_2 - \frac{\partial \Phi_2}{\partial z}, \tag{7}$$

式中 $\gamma = \rho_1/\rho_2$ 且 $0 < \gamma < 1$.

◇ 底部边界条件：

$$\frac{\partial \Phi_2}{\partial z} = 0, \qquad (-\infty < x < +\infty,\ z = -H). \tag{8}$$

匹配条件

连续性：速度和压力

对于自由面区域和板覆盖区域的分界面 $(x=0,\ -H<z<0,\ m=1,2)$，我们有

$$\left.\frac{\partial\Phi_m(x,y,z)}{\partial x}\right|_{x=0^-}=\left.\frac{\partial\Phi_m(x,y,z)}{\partial x}\right|_{x=0^+}, \tag{9}$$

$$\Phi_m(x,y,z)|_{x=0^-}=\Phi_m(x,y,z)|_{x=0^+}. \tag{10}$$

平板的边缘条件 (Edge condition)

假设:

- 自由边假设.

在板的端部 $(x, z) = (0, 0)$ 弯矩和剪切力为零:

$$\frac{\partial^2 \zeta}{\partial x^2} + \nu \frac{\partial^2 \zeta}{\partial y^2} = 0, \qquad (x = 0,\ z = 0), \tag{11}$$

$$\frac{\partial^3 \zeta}{\partial x^3} + (2 - \nu) \frac{\partial^3 \zeta}{\partial x \partial y^2} = 0, \quad (x = 0,\ z = 0). \tag{12}$$

Connection conditions between two elastic plates

Depending on what connecting situation is specified at $(0,0)$, four connection conditions related to

- the deflections,
- rotational angles,
- bending moments, and
- shear forces of the plates

can be found.

A unified representation for various situations can be denoted in a consolidated equation of

$$\boldsymbol{E}^{-}\boldsymbol{\lambda}^{-}=\boldsymbol{E}^{+}\boldsymbol{\lambda}^{+}, \tag{13}$$

where $\boldsymbol{\lambda}^{\pm}=[\zeta(0^{\pm}),\zeta'(0^{\pm}),\zeta''(0^{\pm}),\zeta'''(0^{\pm})]^{\mathsf{T}}$ and $\boldsymbol{E}^{\pm}$ are two 4 by 4 matrixes.

For a given connecting type, $\boldsymbol{E}^{\pm}$ can readily be obtained.

分析问题：应用数学 3+1 步骤

1. 描述 (formulation)：物理问题数学化
2. 求解 (solution)
   - 线性化
   - 垂向有界 $(-h < z < 0)$
   - 边界条件是齐次.

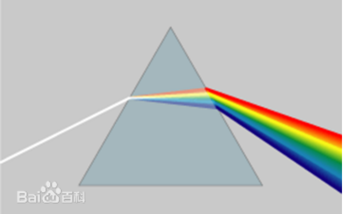
   "数学物理方法"：分离变量法 $\Longrightarrow$

   本征函数展开法 (Fourier 级数法)
   - 本征函数正交性：条件，意义.
3. …… 数学结果物理化 (待续...)

线性问题的求解方法:

1. 色散关系 (dispersion relation)
2. 本征展开 (eigenfunction expansion)

色散（频散）关系 (dispersion relation)



[baidu]

- ♡ 频率 $\omega(k)$：时间周期性
- ♡ 波数 k：　空间周期性
- ♡ 色散关系：相速度 $c(k) = \dfrac{\omega}{k}$
  - 物理上看：时空关系
  - 数学上看：本征关系，解的存在性条件

色散 (频散) 关系: 单层流体[4]

$$\omega^2 = gk\tanh(kh), \tag{14}$$

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kh)}, \tag{15}$$

其中 $k = 2\pi/\lambda$ 是波数 (wave number),

h 为水深, λ 波长 (wavelength).

令 $\delta = kh$. 极限情形: $\delta \ll 1$, $c = \sqrt{gh}$.

♠ 浅水作用 (shoaling):

- 长江后浪推前浪
- ... 落月满屋梁, 犹疑照颜色. 水深波浪阔, 无使蛟龙得.[3]

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[3] 唐朝诗人杜甫的古诗作品《梦李白二首其一》

[4] 刘沛清 (2017),《流体力学通论》, 科学出版社, p. 246

两流体系统中自由表面重力波的色散关系

线性问题的分离变量法:

1. Laplace 方程的通解包含 $e^{i(\alpha x+\beta y)+kz}$，其中 $k^2=\alpha^2+\beta^2$，k 为波数.
2. 代入方程和自由面的边界条件，由解的存在性条件，可得[5]

$$\omega^2=\frac{gk(t_1+t_2)}{2(1+\gamma t_1 t_2)}\left[1+(-1)^{m+1}\sqrt{1-4\varepsilon\frac{t_1t_2(1+\gamma t_1 t_2)}{(t_1+t_2)^2}}\right], \tag{16}$$

其中 $m=1,2$, $t_m=\tanh kh_m$, $\varepsilon=1-\gamma$, $h_2=H-h_1$.

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[5] R. W. Yeung & T. C. Nguyen (1999), Journal of Engineering Mathematics, 35: 85–107.

两流体系统中挠曲重力波的色散关系

由控制方程和板覆盖区的边界条件，由数学上解的存在性条件，可得物理上的色散关系:[6]

$$\omega^2 = \frac{gk(G_1 t_1 + G_2 t_2 + \varepsilon\sigma t_1 t_2)}{2[1+\gamma t_1 t_2 + \sigma(t_1+\gamma t_2)]} \times \left\{ 1 + (-1)^{m+1}\sqrt{1 - 4\varepsilon\frac{G_1 t_1 t_2[1+\gamma t_1 t_2 + \sigma(t_1+\gamma t_2)]}{(G_1 t_1 + G_2 t_2 + \varepsilon\sigma t_1 t_2)^2}} \right\} \tag{17}$$

其中 $\Gamma = k^4 D/(\rho_1 g)$, $\sigma = kd\rho_{\mathrm{e}}/\rho_1 = kM/\rho_1$,

$$G_1 = 1+\Gamma, \qquad G_2 = 1+\gamma\Gamma.$$

当板厚 d 趋向零，式 (17) 退化为式 (16).

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[6] D. Q. Lu* & C. Z. Sun (2013), Journal of Hydrodynamics, 25(3): 339–347, DOI: 10.1016/S1001-6058(11)60372-8.

自由表面重力波

给定一个频率 ω,
自由表面重力波色散方程的根为

1. 两个正实根: k_{0_1}, k_{0_2}
   —— 传播波模态
   - 表面波 k_{0_1}, $m = 1$
   - 界面波 k_{0_2}, $m = 2$,
   - 对于给定频率 ω, 有 $k_{0_1} < k_{0_2}$.
2. 无限多个纯虚根: $\mathrm{i}k_i$
   $i = 1, 2, 3 \ldots$, 其中 k_i 为正实数.
   —— 消散模态

挠曲重力波 / 水弹性波[8]

给定一个频率 ω，

挠曲表面重力波色散方程的根为

1. 两个正实根：κ_{0_1}，κ_{0_2} —— 传播波模态
2. 无限多个纯虚根：$\mathrm{i}\kappa_j$ —— 消散模态
3. 两个额外的项：κ_{I}，κ_{II}，两个复数根，实部为正 —— 板覆盖区的衰减的传播模态

♠ 两个额外的项：数学解的物理解释!?[7]

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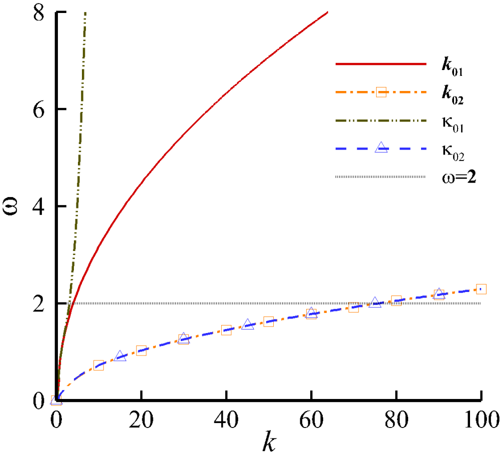
[7] Q. R. Meng & D. Q. Lu* (2017), Journal of Fluids and Structures, 68: 295–309, DOI: 10.1016/j.jfluidstructs.2016.10.014.

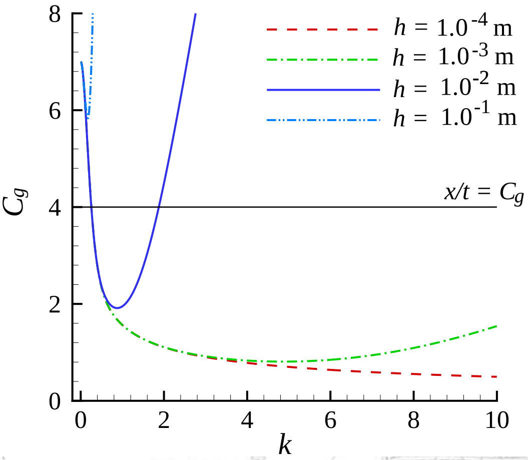
[8] 挠曲重力波 flexural–gravity waves; 水弹性波 hydroelastic waves

色散关系：时空关系，频率（周期）与波数（波长）



ω 与 k，其中 $h = 0.2$ and $\gamma = 0.9$. 显然，k_{0_2} 和 κ_{0_2} 非常接近.

色散关系：群速度与波数 (波长)



本征 (eigen-)

- 本征 (eigen-) 值 vs.
  特征 (characteristic) 值，固有值
- 匹配渐近展开法：[9] matching (匹配)
- 本征函数展开匹配法：[10] patching (拼接)
- 模态展开法[11]
  - 湿模态
  - 干模态

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[9] The method of matched asymptotic expansions

[10] The method of matched eigenfunction expansions;
The eigenfunction expansion–matching method

[11] The mode expansion method

本征函数展开匹配法 (I): 分区 + 展开

分区

将整个流体区域分成:

- 开阔水域: 入射势 + 反射势
- 板覆盖区域: 透射势

展开

利用流体的上表面和底部边界条件,
将每个区域内的速度势展开为:
由本征函数构成的级数形式解.

本征函数展开法

对于开阔水域 $(-\infty < x < 0)$，空间速度势包括两部分 $\phi(x,y,z) = \phi^{\mathrm{I}}(x,y,z) + \phi^{\mathrm{R}}(x,y,z)$.

♣ 单层流体:

$$\phi(x,z) = \left(I_0 \mathrm{e}^{\mathrm{i}k_0 x} + R_0 \mathrm{e}^{-\mathrm{i}k_0 x}\right) Z_0 + \sum_{i=1}^{\infty} R_i \mathrm{e}^{k_i x} Z_i.$$

♠ 两层流体:

$$\phi^{\mathrm{R}}(x,y,z) = \left(R_{0_1} \mathrm{e}^{-\mathrm{i}\alpha_{0_1} x} Z_{0_1} + R_{0_2} \mathrm{e}^{-\mathrm{i}\alpha_{0_2} x} Z_{0_2} + \sum_{i=1}^{\infty} R_i \mathrm{e}^{\alpha_i x} Z_i \right) \times \mathrm{e}^{\mathrm{i}\beta y}. \tag{18}$$

本征函数展开法

板覆盖区 $(0 < x < +\infty)$，透射势

♣ 单层流体：

$$\phi^{\mathrm{T}}(x,z) = T_0 \mathrm{e}^{\mathrm{i}\kappa_0 x} Y_0 + \sum_{j=\mathrm{I}}^{\mathrm{II}} T_j \mathrm{e}^{-\kappa_j x} Y_j + \sum_{j=1}^{\infty} T_j \mathrm{e}^{-\kappa_j x} Y_j.$$

♠ 两层流体：

$$\phi^{\mathrm{T}}(x,z) = \left(T_{0_1} \mathrm{e}^{\mathrm{i}\tilde{\alpha}_{0_1} x} Y_{0_1} + T_{0_2} \mathrm{e}^{\mathrm{i}\tilde{\alpha}_{0_2} x} Y_{0_2} + \sum_{j=\mathrm{I}}^{\mathrm{II}} T_j \mathrm{e}^{-\tilde{\alpha}_j x} Y_j \right.$$

$$\left. + \sum_{j=1}^{\infty} T_j \mathrm{e}^{-\tilde{\alpha}_j x} Y_j \right) \mathrm{e}^{\mathrm{i}\beta y}, \tag{19}$$

两层流体的垂向本征函数 $V(k,z)$

$$\{Z_i(z), Y_{0_m}(z), Y_j(z)\} = \{V(\mathrm{i}k_i, z), V(\kappa_{0_m}, z), V(\mathrm{i}\kappa_j, z)\},$$
$$(i = 1, 2, \dots;\ m = 1, 2;\ j = \mathrm{I}, \mathrm{II}, 1, 2, \dots). \qquad (20)$$

其中

$$V(k,z) = \frac{1}{2\gamma\cosh kH}\Big\{(1+\gamma)\cosh k(H+z) + \varepsilon\Big[\cosh k(h_1-h_2+z) + \frac{gk}{\omega^2}\Big(\sinh k(h_1-h_2+z) - \sinh k(H+z)\Big)\Big]\Big\}, \quad (-h_1<z<0), \qquad (21)$$

$$V(k,z) = \frac{\cosh k(H+z)}{\cosh kH}, \qquad (-H<z<-h_1). \qquad (22)$$

本征函数展开匹配法 (II): 如何确定本征函数的系数 — 内积法

通常, 利用

1. 函数的“内积”

$$< Z_i(z), Z_j(z) >= \int_a^b Z_j(z)Z_j(z)\mathrm{d}z. \qquad (23)$$

2. 本征函数的正交性
   — 例如 Sturm–Liouville 本征值问题中的广义 Fourier 级数.

问题: 然而, 板覆盖区域的垂向本征函数在通常的内积意义下并不正交!

数学模型：流固耦合

根源：频域上流固耦合的边界条件：

$$-\rho_1\omega^2\phi_1 + \left(\rho_1 g + D\nabla^4 - M\omega^2\right)\frac{\partial\phi_1}{\partial z} = 0. \quad (24)$$

方程 (24) 中的微分算子不是“自伴随”/“自共轭” (self-adjoint)!!!

因此，其本征函数族不具有正交性.

注：对于 $\forall u(x), v(x) \in C^2(a,b)\bigcap\mathcal{L}^2[a,b]$,
微分算子 $\mathcal{L}[v(x)]$ 是自伴随的，当

$$\int_a^b u(x)\mathcal{L}[v(x)]\mathrm{d}x = \int_a^b v(x)\mathcal{L}[u(x)]\mathrm{d}x.$$

本征函数展开匹配法 (II)

出路:

匹配

- 共轭梯度法. Fox 和 Squire (1994).
- 内积法. Sahoo 等 (2001), Teng 等 (2001).

Sahoo 等 (2001) 提出了一种新的内积定义, 但是

- 数学上引入不必要的复杂性.
- 计算效率并未有实质性的提高.

因此，运用本征函数展开法时，我们**提出**

- 一种新的算法:
  直接利用开阔水域内的垂向本征函数.
- 一种新的内积定义.[12]

路线:

- 新算法，老问题：验证算法的有效性和可靠性
  将新的算法先应用到已有的研究内容 (单层流体的情形).[13]
- 新算法，新问题......

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[12] F. Xu & D. Q. Lu* (2010), International Journal of Engineering Science, 48(9): 408–419, DOI: 10.1016/j.ijengsci.2010.04.007.

[13] F. Xu & D. Q. Lu* (2009), Journal of Hydrodynamics, 21(4): 526–530, DOI: 10.1016/S1001-6058(08)60180-8.

本征函数展开匹配法 (III): 截断

作用内积

开阔水域内的垂向本征函数族：在**通常的**函数内积意义下具有正交性

$$< Z_i(z), Z_l(z) >= \int_{-h}^{0} Z_i(z)Z_l(z)\mathrm{d}z. \tag{25}$$

将每个本征函数在匹配方程两端“作用”内积，得到一组代数方程.

截断求解

将上面得到无限维代数方程组以及两个板的的端部边界条件方程适当截断，得到一个有限维线性方程组，求解得到展开系数.

垂向本征函数的内积：两层

为了得到展开式未知系数的联立方程，我们对两层流体引入了垂向本征函数的内积定义式如下[14]

$$\begin{aligned} P_{nl} &= \ < Z_n(z), Z_l(z) > \\ &= \int_{-H}^{-h_1} Z_n(z) Z_l(z) \mathrm{d}z + \gamma \int_{-h_1}^{0} Z_n(z) Z_l(z) \mathrm{d}z, \\ & \quad (n, l = 0_1, 0_2, 1, 2, \ldots). \end{aligned} \tag{26}$$

满足函数的正交性：

$$P_{nl} = 0, \qquad (n \neq l), \tag{27}$$

$$P_{nn} \neq 0, \qquad (n = l). \tag{28}$$

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[14] F. Xu & D. Q. Lu* (2010), International Journal of Engineering Science, 48(9): 408–419, DOI: 10.1016/j.ijengsci.2010.04.007.

垂向本征函数的内积：两层

对于板覆盖区，

$$Q_{qn}=\int_{-H}^{-h_1} Y_q(z)Z_n(z)\mathrm{d}z+\gamma\int_{-h_1}^{0} Y_q(z)Z_n(z)\mathrm{d}z,$$
$$(q=0_1,0_2,\mathrm{I},\mathrm{II},1,2,\ldots;\ n=0_1,0_2,1,2,\ldots). \tag{29}$$

正交性：

$$Q_{nl}\neq 0, \qquad (n=l)\ \&\ (n\neq l). \tag{30}$$

垂向本征函数的内积：三层

对于三层流体，我们引入了垂向本征函数的内积定义式如下[15]

$$
\begin{aligned}
P_{ln} &= \langle Z_l, Z_n \rangle \\
&= \int_{-H_3}^{-H_2} Z_l \cdot Z_n \,\mathrm{d}z + \gamma_2 \int_{-H_2}^{-H_1} Z_l \cdot Z_n \,\mathrm{d}z \\
&\quad + \gamma_1\gamma_2 \int_{-H_1}^{0} Z_l \cdot Z_n \,\mathrm{d}z, \\
&\quad (l, n = 0_1, 0_2, 0_3, 1, 2, 3, \cdots).
\end{aligned} \tag{31}
$$

满足函数的正交性，其中 $\gamma_1 = \rho_1/\rho_2$, $\gamma_2 = \rho_2/\rho_3$.

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[15] Q. R. Meng & D. Q. Lu* (2017), Applied Mathematics and Mechanics – English Edition, 38(4): 567–584, DOI: 10.1007/s10483-017-2185-6.

N plates on a M-layer fluid[16]

The definition of the inner product for the M-layer fluid is newly defined by

$$\langle U, V \rangle = \sum_{m=1}^{M} \frac{\rho_m}{\rho_M} \int_{-H_m}^{-H_{m-1}} U \cdot V \, \mathrm{d}z, \tag{32}$$

where $U(z)$ and $V(z)$ represent arbitrary vertical eigenfunctions, and $H_0 = 0$ for $m = 1$

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[16] Q. R. Meng & D. Q. Lu* (2018), European Journal of Mechanics B/Fluids, 67: 329–340, DOI: 10.1016/j.euromechflu.2017.09.010.

N plates on a M-layer fluid

For the two sort of vertical eigenfunctions, we can derive an orthogonal relation by adding an explicit differential term, namely

$$\left\langle Z_p, \widetilde{Z}_{n,q} \right\rangle - \mathcal{D}_n(p,q) = 0, \tag{33}$$

$$(p = 0_1, 0_2, \cdots, 0_M, 1, 2, \cdots; \tag{34}$$

$$q = 0_1, 0_2, \cdots, 0_M, \mathrm{I}, \mathrm{II}, 1, 2, \cdots),$$

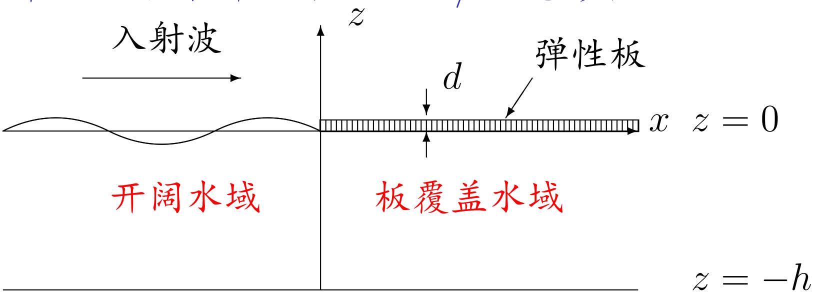
where

$$\mathcal{D}_n(p,q) = \frac{(D_n \widetilde{k}_{n,q}^4 - M_n\omega^2)}{\rho_2\omega^2(k_p^4 - \widetilde{k}_{n,q}^4)} \left[\frac{\partial^3 Z_p}{\partial z^3}\frac{\partial \widetilde{Z}_{n,q}}{\partial z} + \frac{\partial Z_p}{\partial z}\frac{\partial^3 \widetilde{Z}_{n,q}}{\partial z^3} \right]_{z=0},$$

$$(n = 1, 2, \cdots, N). \tag{35}$$

研究内容

[I] 单层流体 + 半无限长板[17]/ 任意形状板[18]



- 运用本征函数展开法时，提出一种新的算法；
- 将新算法先应用到老问题 [T. Sahoo et al (2001), B. Teng et al (2001)] 以验证算法，再应用到新问题.

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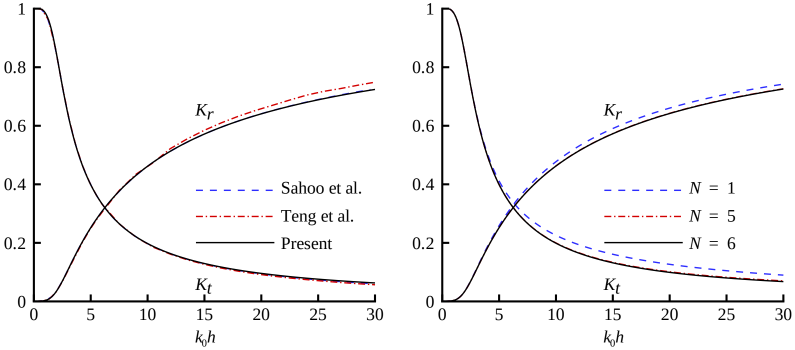
[17] F. Xu & D. Q. Lu* (2009), Journal of Hydrodynamics, 21(4): 526–530, DOI: 10.1016/S1001-6058(08)60180-8.

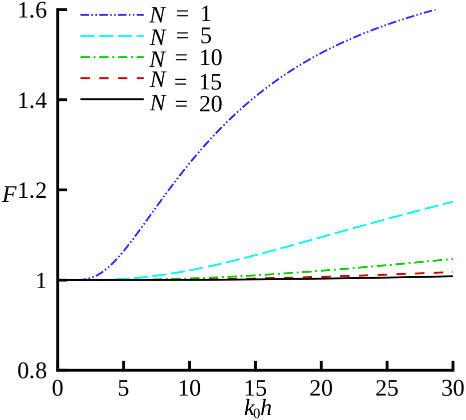
[18] F. Xu & D. Q. Lu* (2011), SCIENCE CHINA Physics, Mechanics & Astronomy, 54(1): 59–66, DOI: 10.1007/s11433-010-4199-3.

数值结果 (I): 反射系数和透射系数



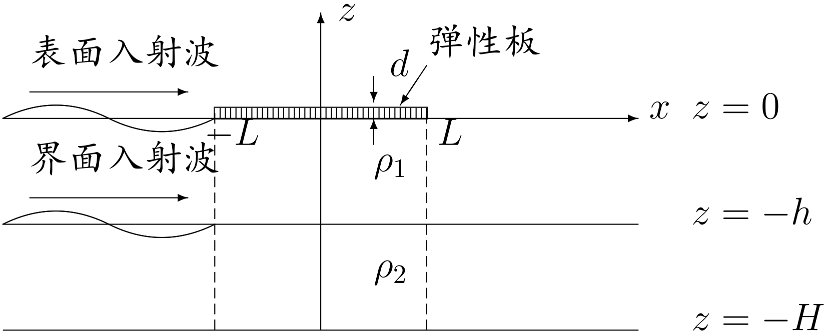
- 与前人的结果一致.
- 就计算反射系数和透射系数而言，我们的方法的计算效率要比他们的都要高一些.

数值结果 (I): 能量守恒关系



随着 k_0h 的增大，解的收敛速度变慢.

[II] 两层流体 + 半无限长板 / 有限长板[19] / 圆板[20]



- 提出了一种新的内积定义.

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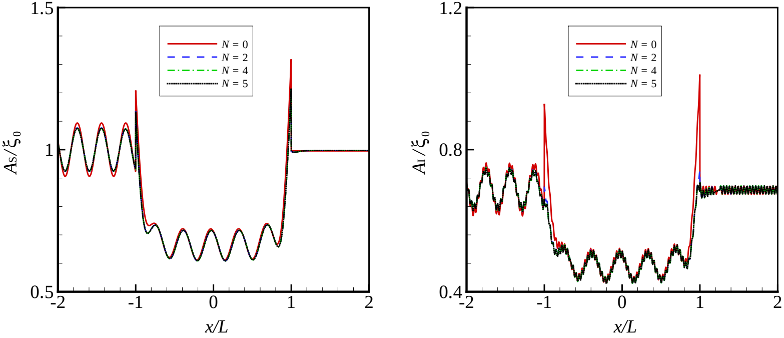
[19] F. Xu & D. Q. Lu* (2010), International Journal of Engineering Science, 48(9): 408–419, DOI: 10.1016/j.ijengsci.2010.04.007.

[20] Q. Lin, D. Q. Lu* & R. W. Yeung (2014), China Ocean Engineering, 28(5): 671–687, DOI: 10.1007/s13344-014-0053-0.

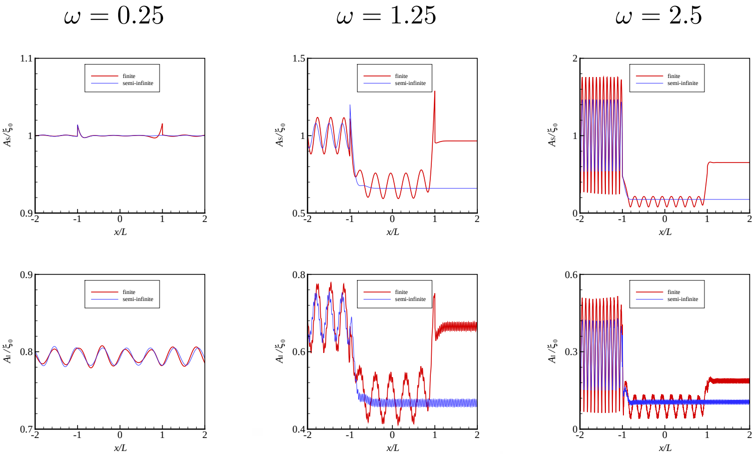
解的收敛性



$$A_{\mathrm{S}}(x) = \left\| \frac{\mathrm{i}}{\omega} \frac{\partial \phi(x,0)}{\partial z} \mathrm{e}^{-\mathrm{i}\omega t} \right\| = \frac{1}{\omega} \left\| \frac{\partial \phi(x,0)}{\partial z} \right\|, \tag{36}$$

$$A_{\mathrm{I}}(x) = \left\| \frac{\mathrm{i}}{\omega} \frac{\partial \phi(x,-h)}{\partial z} \mathrm{e}^{-\mathrm{i}\omega t} \right\| = \frac{1}{\omega} \left\| \frac{\partial \phi(x,-h)}{\partial z} \right\|. \tag{37}$$

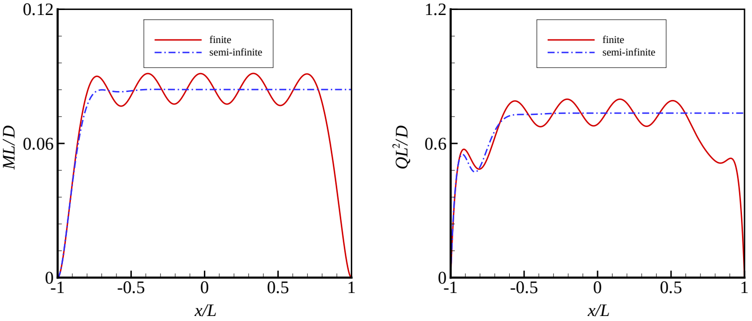
不同入射频率下波的散射与板的响应



$\omega = 0.25$ $\omega = 1.25$ $\omega = 2.5$

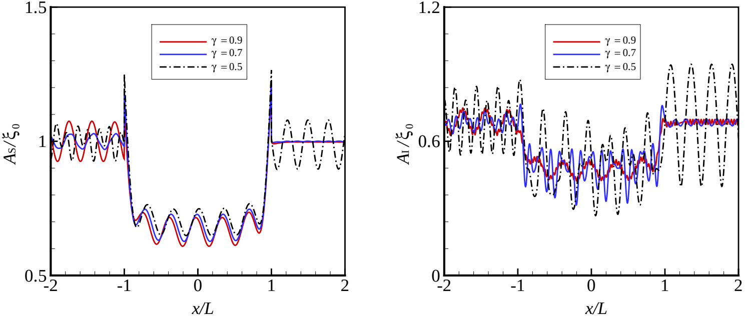
随着频率的增大，波浪的反射增强而透射减弱。

弯矩和剪力



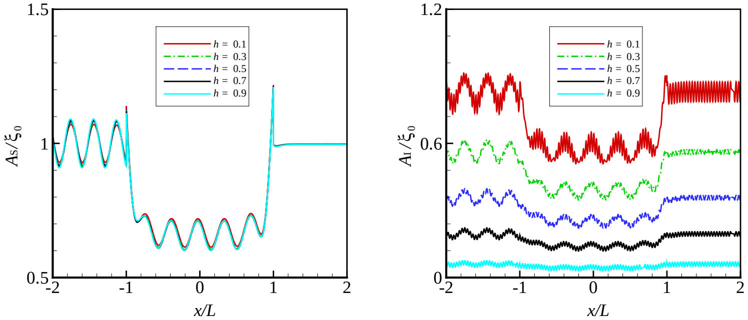
- 端部的弯矩和剪力均为零，与假设一致.
- 远离端部，半无限板：保持不变；有限长板：周期性变化.

密度比



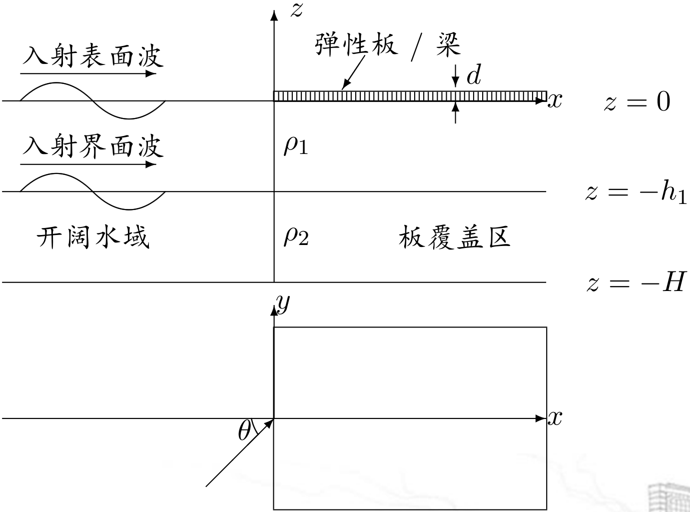
随着密度比的增大，表面和界面位移幅度均减小.

深度比



- 随着深度比的增大，界面位移幅度均减小，而表面位移幅度几乎保持不变.

[III] 两层流体 + 半无限长板 + 斜入射[21]



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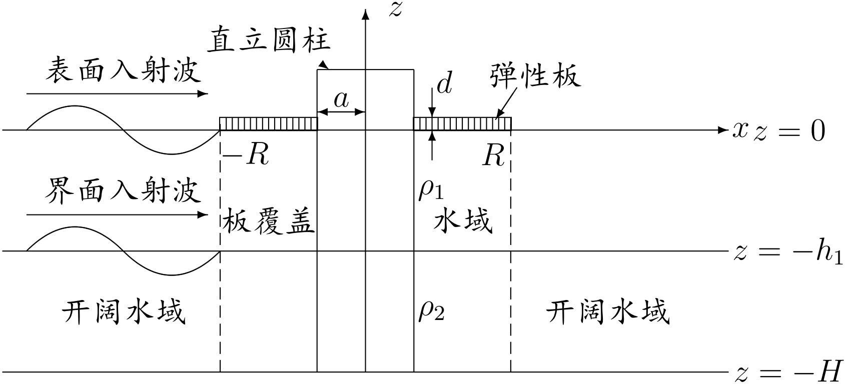
[21] Q. Lin & D. Q. Lu* (2013), Applied Ocean Research, 43: 71–79, DOI: 10.1016/j.apor.2013.07.009.

[IV] 两层流体 + 板环绕直立圆柱[22]



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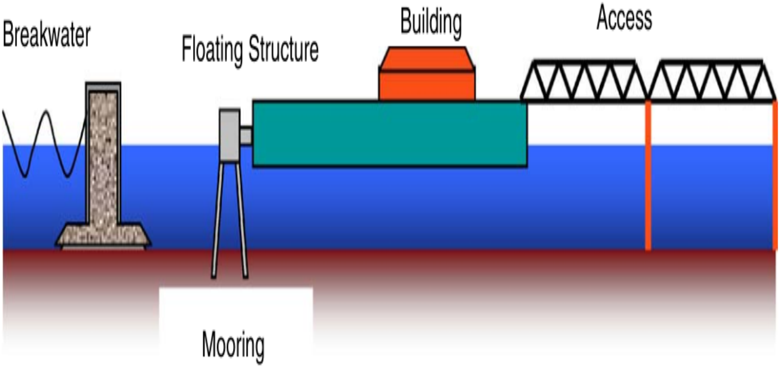
[22] Q. Lin & D. Q. Lu* (2014), European Journal of Mechanics B/Fluids, 44: 10–21, DOI: 10.1016/j.euromechflu.2013.11.004.

A rigid body connected with VLFS[23]



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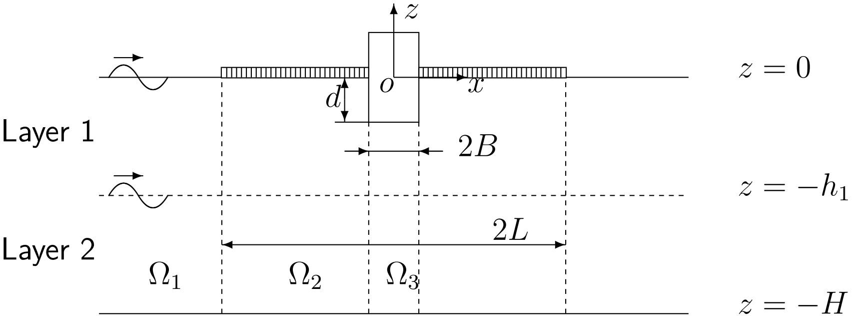
[23] H. Suzuki (2005), "Overview of megafloat: Concept, design criteria, analysis, and design," Marine Structures, 18(2): 111–132.

[V] A semi-immersed rigid body connected with elastic plates in a two-layer fluid[24]



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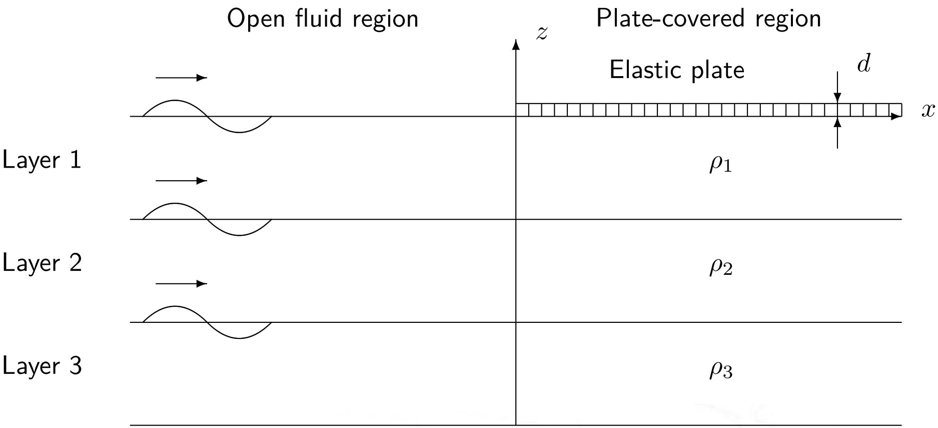
[24] Q. R. Meng & D. Q. Lu* (2017), Journal of Fluids and Structures, 68: 295–309, DOI: 10.1016/j.jfluidstructs.2016.10.014.

[VI] A thin elastic plate floating on a three-layer fluid[25]



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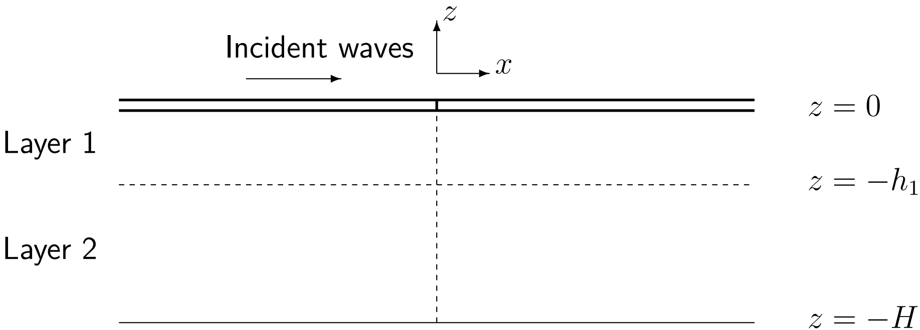
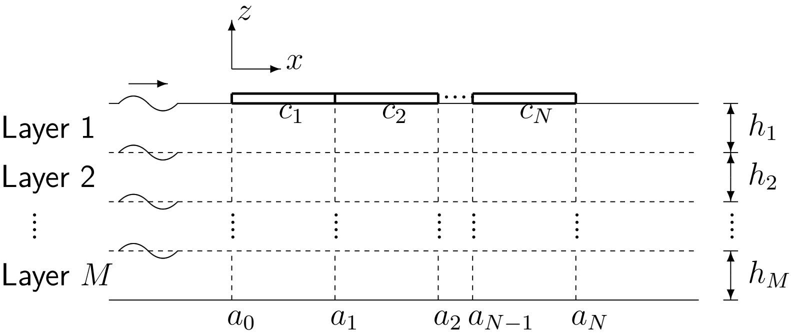
[25] Q. R. Meng & D. Q. Lu* (2017), Applied Mathematics and Mechanics - English Edition, 38(4): 567–584, DOI: 10.1007/s10483-017-2185-6.

[VII] Reflection and transmission of flexural–gravity waves



Incident flexural–gravity waves across the boundary between the domains covered by diverse elastic plates

[VII] N plates floating on M fluid layers[26]



We consider a generalized situation that N finite elastic plates with variable properties are floating on a M-layer fluid, which can be seen as a multi-module very large floating structure (VLFS) on the stratified ocean.

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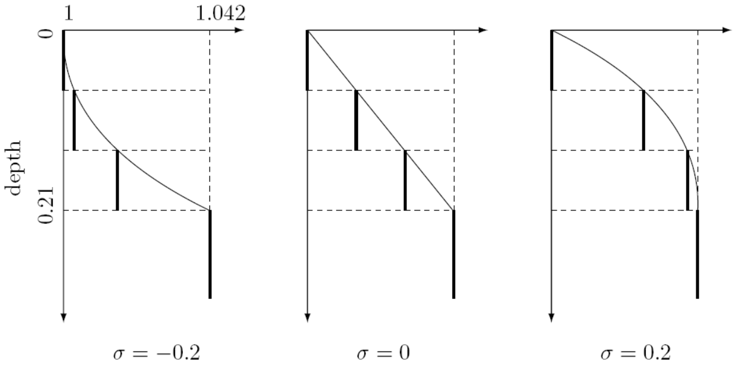
[26] Q. R. Meng & D. Q. Lu* (2018), European Journal of Mechanics B/Fluids, 67: 329–340, DOI: 10.1016/j.euromechflu.2017.09.010.

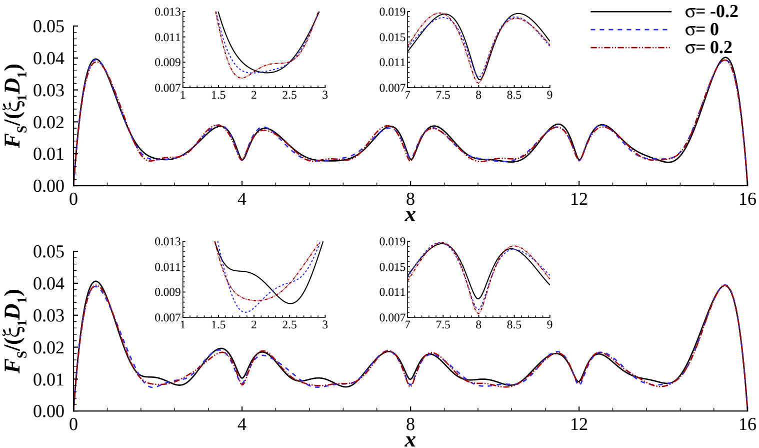
The density ρ_m versus the depth H_{m-1} $(m = 1, \cdots, M)$ follows the parabolic function as follows

$$\rho_m = -4.76\sigma H_{m-1}^2 + (\sigma + 0.2)\, H_{m-1} + 1, \quad (-0.2 \le \sigma \le 0.2)\,,$$

where the parameter σ is employed to simulate the profile of the curve.



Discrete density distribution for a 4-layer stratification.



Amplitude of shear force affected by different density distributions in a (a) 4-layer fluid, (b) 8-layer fluid

- The above figures show the calculation results for the amplitudes of shear force.
- A conspicuous variation is exhibited for the values along the whole structure, especially at the middle area of every single plate and the neighborhoods nearby the connections, which has been plotted in the subgraphs.
- This phenomenon becomes more intense in the 8-layer fluid, which implies the fluid stratification will generate essential impact on the inner shear forces of the floating elastic plates.
- Investigations on a more refined fluid stratification are necessarily important.

思考与延伸

漂浮板问题，最近考虑[27]

1. 海流的作用
2. 板的侧向应力

漂浮板问题：

- 表面和界面以同样频率传播波动的局限性
- 吃水为零?

方法的延伸到复杂结构：海底多孔障碍物[28][29]

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[27] D. Q. Lu (2014), Journal of Hydrodynamics, 26(2): 339–341, DOI: 10.1016/S1001-6058(14)60037-8.

[28] Q. R. Meng & D. Q. Lu* (2016), Journal of Hydrodynamics, 28(3): 519–522, DOI: 10.1016/S1001-6058(16)60656-X.

[29] Q. Lin, Q. R. Meng & D. Q. Lu* (2018), Journal of Hydrodynamics, 30(3): 453–462, DOI: 10.1007/s42241-018-0041-6.

总结

研究方法上: 提出了

1. 一种新的算法, 运用开阔水域的本征函数
2. 一种适合于强分层流体的内积定义

研究内容上:

1. 内界面波和表面波相互耦合作用及其对表面弹性波的动力作用

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